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Stabilizing a breather in the damped nonlinear Schrödinger equation driven by two frequencies

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In the framework of adiabatic perturbation theory and energy balance conditions, we analyze the possibility of stabilizing a breatherlike state (a bound state of two solitons with coinciding centers) in the damped (1+1)-dimensional cubic nonlinear Schrödinger equation by a two-frequency ac drive. The analytic predictions for the threshold of driving strengths above which a stable breather exists are in good agreement with those measured in numerical simulations.

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The damped, ac-driven cubic nonlinear Schrödinger (NLS) equation,

$$iu_t + u_{xx} + 2|u|^2u = -i\alpha u + \epsilon e^{i\omega t}, \tag{1}$$

is a fundamental model describing a number of nonlinear dynamical systems in solid-state physics, plasma, optics, etc. (see, e.g., the review paper [1]). It is well known that, if the frequency ω is positive and the amplitude of driving ϵ exceeds the threshold value

$$\epsilon_{\text{thr}} = \frac{2}{\pi}\alpha\sqrt{\omega}, \tag{2}$$

which is proportional to the dissipative constant α, Eq. (1) admits stable localized solutions in the form of a soliton with the internal frequency ω phase-locked to the driving frequency [2]:

$$u(x,t) = \eta \operatorname{sech}(\eta x) e^{i(\omega t - \phi)}, \tag{3}$$

where $\eta = \sqrt{\omega}$ and

$$\epsilon \sin\phi = \frac{2\alpha}{\pi}\eta.$$

Strictly speaking, the solution is the above soliton riding on top of a small amplitude oscillating background. This result was obtained analytically by means of perturbation theory [2], which is applicable provided $\alpha \ll \omega$ and $\epsilon \ll \omega^{3/2}$. At sufficiently large ϵ, the dynamics of the perturbed NLS system governed by Eq. (1) becomes very complicated. This complexity may be summarized as the onset of low-dimensional dynamical chaos [3].

The unperturbed NLS equation, being exactly integrable, admits not only the exact one-soliton, but also multisoliton solutions. In particular, the initial condition,

$$u_0(x) = N\eta \operatorname{sech}(\eta x), \tag{4}$$

where N is a positive integer, gives rise to the so-called breather, or N-soliton solutions, which may be regarded as a nonlinear superposition of N different solitons with coinciding centers. A fundamental difference between the N-soliton breather and the fundamental ($N = 1$) soliton is the fact that the breather has *shape* oscillations, i.e., $|u(x)|$ is a periodic function of time. In particular, for the case of $N = 2$, the analytical solution is [4]

$$u(x,t) = \frac{4[\cosh(3x) + 3e^{i8t}\cosh(x)]}{\cosh(4x) + 4\cosh(2x) + 3\cos(8t)} e^{it}, \tag{5}$$

for $\eta = 1$. Due to the scaling property of the NLS, we can always rescale η to 1. This breather solution has a carrier-wave frequency $\omega_c = 1$ and shape-mode frequency $\omega_s = 8\omega_c$.

From numerical simulations of Eq. (1), it is well known that the perturbed NLS equation cannot support any type of breathers but the fundamental soliton. Analytically, perturbation-induced evolution of an $N = 2$ breather was considered in Ref. [5]. It has been demonstrated that even under the action of purely Hamiltonian perturbations, which are always much gentler than the dissipative perturbations in Eq. (1), the breather degrades into a single-soliton state through emission of radiation. In the dissipatively perturbed model, the degradation will be much faster.

Nevertheless, one can expect that existence of a breather in a damped system may be supported by a *two-frequency* drive. In the simplest case, this type of drive can be described by the equation

$$iu_t + u_{xx} + 2|u|^2u = -i\alpha u + \epsilon_1 e^{i\omega_1 t} + \epsilon_2 e^{i\omega_2 t}. \tag{6}$$

Notice that, as long as $\omega_1 \neq \omega_2$, the phase difference between two drives can always be removed by a translation in time and a trivial gauge transformation in $u(x,t)$. It should be noted that dissipative models with multifrequency drivers were considered earlier [6, 7] in a parameter region where the amplitudes ϵ_n, $n = 1, 2$, were sufficiently large in order to analyze the onset of spatiotemporal chaos in these models. Here, we are interested in the case of ϵ_n sufficiently small, where dynamical chaos is

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not expected, but the two-frequency driver with appropriate values of its frequencies and amplitudes should be able to stabilize a breather in a form close to the breather of the unperturbed NLS system [Eq. (5)].

In the following, we analyze this possibility of locking a breatherlike state with the two-frequency driver. The unperturbed NLS has the energy,

$$H = \int (|u_x|^2 - |u|^4) dx,$$

and the norm,

$$N = \int |u|^2 dx,$$

which are both conserved. For the perturbed NLS, Eq. (6), we have

$$\frac{dH}{dt} = -2\alpha \int_{-\infty}^{+\infty} dx\, \mathrm{Im}\{u u_t^*\} - 2\int_{-\infty}^{+\infty} dx\, \mathrm{Re}\{f^* u_t\},$$
$$\frac{dN}{dt} = -2\alpha N + 2\int_{-\infty}^{+\infty} dx\, \mathrm{Im}\{f u^*\}, \tag{7}$$

where $f = \epsilon_1 e^{i\omega_1 t} + \epsilon_2 e^{i\omega_2 t}$. In the following adiabatic perturbation calculation, we use the ansatz that the breather solution in Eq. (6) assumes the form

$$u(x,t) = \frac{4[\cosh(3x) + 3e^{i(\omega_s t + \phi_1 - \phi_2)}\cosh(x)]}{\cosh(4x) + 4\cosh(2x) + 3\cos(\omega_s t + \phi_1 - \phi_2)} \times e^{i(\omega_c t - \phi_1)}, \tag{8}$$

where $\omega_c = 1$, $\omega_s = 8\omega_c$. By noting the structure of the breather [Eq. (8)], we choose $\omega_1 = \omega_c$ to phase-lock the carrier-wave part of Eq. (5) and $\omega_2 - \omega_1 = \omega_s$ to phase-lock the shape mode. Furthermore, phase-locking demands that the energy and the norm be balanced in one period of shape oscillation, i.e., there are no net changes during one period [8, 9]:

$$\left\langle \frac{dH}{dt} \right\rangle = 0,$$
$$\left\langle \frac{dN}{dt} \right\rangle = 0, \tag{9}$$

where $\langle\ \rangle$ is the time average over one period $T = 2\pi/\omega_s$.

From Eq. (9), a straightforward calculation leads to

$$14\alpha = I_1 \epsilon_1 \sin\phi_1 + I_2 \epsilon_2 \sin\phi_2,$$
$$18\alpha = 9 I_1 \epsilon_1 \sin\phi_1 + I_2 \epsilon_2 \sin\phi_2, \tag{10}$$

where

$$I_1 = \int_{-\infty}^{+\infty} dx \frac{3\cosh(x)[\sqrt{1-a(x)^2} - 1] + \cosh(3x) a(x)}{3\sqrt{1-a(x)^2}} \approx 0.950\,45,$$

$$I_2 = \int_{-\infty}^{+\infty} dx \frac{3\cosh(3x)[\sqrt{1-a(x)^2} - 1] + 9\cosh(x) a(x)}{\sqrt{1-a(x)^2}} \approx 7.0582,$$

$$a(x) = \frac{3}{\cosh(4x) + 4\cosh(2x)}.$$

We can readily conclude from Eq. (10) that

$$\epsilon_1 \sin\phi_1 = \frac{\alpha}{2I_1},$$
$$\epsilon_2 \sin\phi_2 = \frac{27\alpha}{2I_2}, \tag{11}$$

and the thresholds for the strength of the drivers are $\epsilon_{1,\mathrm{thr}} = \alpha/(2I_1) \approx 0.5261\alpha$, $\epsilon_{2,\mathrm{thr}} = 27\alpha/(2I_2) \approx 1.9127\alpha$, below which a breather state is not sustainable by the two-frequency drive. We note that for the fundamental soliton, the solution has the structure $u(x,t) = u(x)\exp(i\omega t)$. With this ansatz, the rate of change of the energy is related to that of the norm by

$$\frac{dH}{dt} = -\omega \frac{dN}{dt}.$$

Hence, to obtain the threshold in Eq. (2), it suffices to consider only one of the equations in Eq. (7) in the adiabatic perturbation theory.

We have performed direct numerical simulations for the model, Eq. (6), on a sufficiently long system with periodic boundary conditions. The functional form, Eq. (4), was used as the initial condition. Figure 1 shows a locked breather state. From the above perturbation theory, the locked breather states do not exist in the region $\epsilon_1 \le 0.0526$, and $\epsilon_2 \le 0.191$ for $\alpha = 0.1$. Figure 2 shows that the thresholds measured from the numerical simulations (crosses), and the theoretical predictions (solid lines) are in excellent agreement. For $\epsilon_1 \ge 0.14$, along the direction $\epsilon_2 \approx 0.19$, we did not observe locked breather states. For $\epsilon_2 \ge 0.35$, along the direction $\epsilon_1 \approx 0.053$, we observed some windows of ϵ_2 in which it is still possible to lock a breather state. Even for these breather states which are significantly deformed away from the unperturbed breather form, the threshold estimates still hold very well. Those initially breathing states which are outside the locking region generally decay to fundamental soliton(s). They may further decay to a radiation background and eventually die out. In addition to

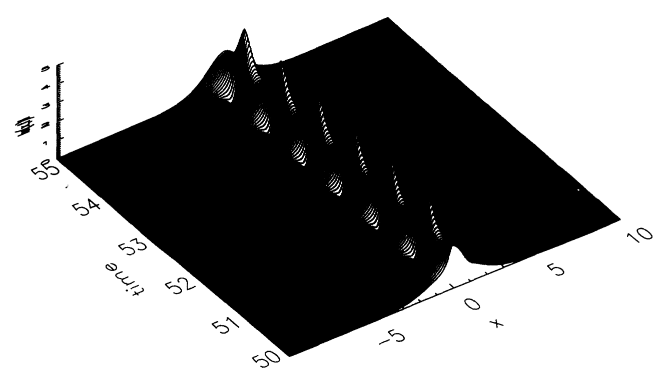

FIG. 1. A breather stabilized by a two-frequency drive: $\omega_1 = 1$, $\omega_2 = 9$, $\alpha = 0.1$. Plotted here is $|u(x,t)|$.

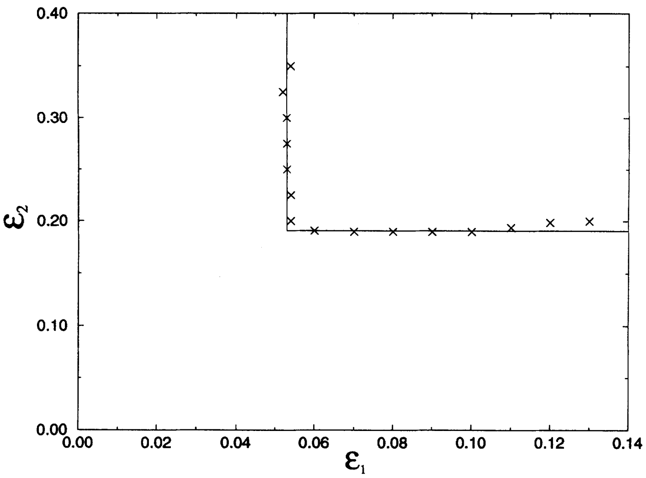

FIG. 2. Thresholds for locking a breather state with $\alpha = 0.1$ (see text). "$\times$," from numerical simulations; solid lines, the theoretical predictions.

$\omega_2 = 9$, for $\omega_2 = 8$ and $\omega_2 = 10$, we have also observed locked breather states with the initial condition Eq. (4). This implies that the locking phenomenon is more general than merely locking to the exact solution [Eq. (5)] adiabatically. For frequencies far from $\omega_2 = 9$, we were not able to lock any breathing states with the initial condition Eq. (4).

In summary, we have demonstrated that the two-frequency drive with properly selected frequencies and sufficiently large amplitudes is able to support breather-like states in the damped NLS system. This observation clearly demonstrates that, even far from the onset of dynamical chaos, the dynamics of simple models like that based on Eq. (6) can be complicated. As for physical realizations of the effect considered, it can manifest itself through measurements of the energy absorption rate in systems described by the damped NLS system (charge density waves [2], etc.) driven by a superposition of two ac fields with different frequencies. To derive the thresholds for sustaining these breather states, we have applied the adiabatic perturbation theory and energy-balance arguments to the energy and the norm of the system. It is in contrast to the case of locking one-soliton states, where only one of them is needed. Direct numerical simulations showed that the analytic predictions for the thresholds obtained this way agree very well with the observed ones.

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[1] Yu. S. Kivshar and B.A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).
[2] D.J. Kaup and A.C. Newell, Phys. Rev. B **18**, 5162 (1978).
[3] For example, A.R. Bishop, R. Flesch, M.G. Forest, D.W. McLaughlin, and E.A. Overman, SIAM J. Math. Anal. **21**, 1511 (1990), and references therein.
[4] J. Satsuma and N. Yajima, Suppl. Prog. Theor. Phys. **55**, 284 (1974).
[5] B.A. Malomed, Phys. Lett. A **154**, 441 (1991).
[6] K. Nozaki and N. Bekki, Phys, Rev. Lett. **50**, 1226 (1983).
[7] K. Nozaki and N. Bekki, Phys. Lett. A **102**, 383 (1984).
[8] P.S. Lomdhal and M.R. Samuelsen, Phys. Rev. A **34**, 664 (1986).
[9] P.S. Lomdhal and M.R. Samuelsen, Phys. Lett. A **128**, 427 (1988).

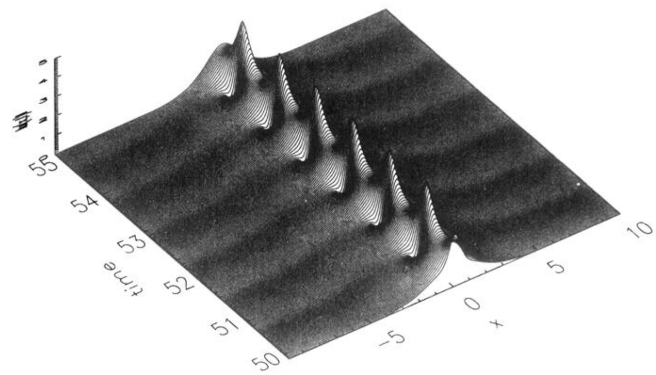

FIG. 1. A breather stabilized by a two-frequency drive: $\omega_1 = 1$, $\omega_2 = 9$, $\alpha = 0.1$. Plotted here is $|u(x,t)|$.